Journal of Engineering Physics and Thermophysics, Vol. 80, No. 5, 2007

ON THE THEORY OF THIN BODIES

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It is shown that, in accordance with the law of plane sections representing the basis of the theory of thin bodies, two analogies are true: a nonstationary analogy and a stationary one. Within the framework of the stationary analogy, a new-type expansion reducing a three-dimensional problem to a stationary two-dimensional one is introduced. The asymptotic conditions of validity of the stationary and nonstationary analogies were determined and the boundary-layer conceptions for both cases were compared. Solutions of the internal and external problems on a two-dimensional flow in a narrow zone were obtained in the closed form. An asymptotically correct mathematical model of a flow in a film is proposed. The stationary and nonstationary analogies for a swirling flow around a body and a flow around a rotating body were determined.

Introduction. In the theory of thin bodies, an ideal-fluid flow in a thin zone is considered. The range of applications of this theory is fairly wide: flows around airplanes and rockets, hydroplaning of an elongated surface, motion of an elongated vessel, swimming of fishes, internal fluid flows in channels and blood vessels, evolution of a column-like vortex, and migration of elongated particles in a flow.

It is assumed in the classical theory of thin bodies that the direction of the velocity-vector of an undisturbed flow u_∞ is known and a disturbed motion occupies a narrow region around a body. The use of a series expansion parameter — the dimensionless thickness of this zone — opens up wide possibilities for application of asymptotic methods.

The development of the theory of thin bodies, defining a stationary flow, was initiated by the Munk investigations on the calculation of a dirigible (see [1]). Von Kármán and Tsien extended the Munk theory to the supersonic flight speeds. Then this theory was used by many researchers for calculating the aerodynamic characteristics of differently shaped bodies, airfoils, and aerodynamic combinations. Of special interest are works of Jones who extended the theory to the case of separation flows, including vortex sheets, and works of Ward who refined the method of calculating the bodies around which a supersonic stream flows. The use of asymptotic methods made it possible to simplify and formalize the theory of thin bodies [2–4]. A nonstationary analogy was also determined for a transonic flow [5].

If a region is extended along the x axis and its thickness λ is small, the following relations are true for its interior part, where $y, z = O(\lambda l)$ and $x = O(l)$:

$$\left|\frac{\partial}{\partial x}\right| = O\left(\frac{1}{l}\right), \quad \left|\frac{\partial}{\partial y}\right| \sim \left|\frac{\partial}{\partial z}\right| = O\left(\frac{1}{\lambda l}\right).$$

Consequently, in the first approximation, the derivative $\partial/\partial x$ is small as compared to $\partial/\partial y$ and $\partial/\partial z$. Analogous estimations are also true for the higher derivatives. If the initial equations are symmetric relative to the variables x, y, and z as, for example, the Laplace, Maxwell, and Stokes equations, longitudinal derivatives are absent in them in the first approximation. In this case, the law of plane sections is valid: the equations in the plane x = const become two-dimensional and the "frozen" coordinate x denotes only the cross section selected. This is the essence of the law of plane sections, on which the theory of thin bodies is based.

In the Euler equation, as in the Navier–Stokes equations, the x derivative enters nonlinearly in the inertial term containing the substantive derivative

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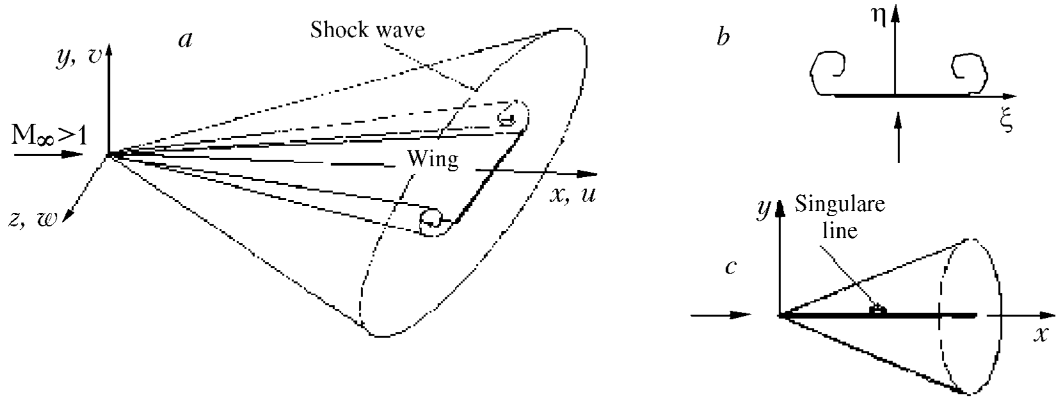


Fig. 1. Scheme of a supersonic flow around a triangular wing of low aspect ratio ($\alpha << 1$): a) general view; 2) interior part of the region (ξ, $\eta \sim 1$); c) exterior part of the region (y, $z \sim 1$).

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

In the classical formulation, where the following expansions are true:

$$u(x, y, z; \lambda) = u_\infty + \lambda^2 U(x, \eta, \xi) + o(\lambda^2), \quad v(x, y, z; \lambda) = \lambda V(x, \eta, \xi) + o(\lambda),$$

$$w(x, y, z; \lambda) = \lambda W(x, \eta, \xi) + o(\lambda), \quad y = \lambda\eta, \quad z = \lambda\xi, \tag{1}$$

and the x coordinate is like to the time coordinate:

$$x = u_\infty t, \quad \frac{d}{dt} \approx \frac{\partial}{\partial t} + V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial \xi}, \tag{2}$$

an analogy between stationary three-dimensional and nonstationary two-dimensional flows applies to the subsonic, supersonic, and hypersonic regimes [4–6]. Figure 1 shows schemes of a supersonic flow around a low-aspect-ratio triangular wing with a small angle of attack, separating from its side edges. Expansion (1) and the nonstationary analogy are valid in the interior part of the region being considered ($x \sim l$, $y \sim \lambda l$, $z \sim \lambda l$), where the whole separation is located. In the exterior part of the region (x, y, $z \sim l$), the total Euler equations are valid and the whole disturbed region is reduced to the singular line $y = z = 0$.

The law of plane sections can be used in various forms. For example, in the Prandtl–Van Dyke theory of high-aspect-ratio wings, sections parallel to the direction of an undisturbed flow are selected, and, in the theory of finite-aspect-ratio rectangular wings positioned at a small angle of attack, sections normal to the side edge of a wind are considered [7].

Stationary Analogy. The stationary analogy defined by expansion (2) corresponds to the case of *weak interaction* of a flow around a body with this body; in this case, the flow velocity is near-disturbed, and the angles of inclination of the streamlines to the x axis are small, i.e., $v/u_\infty = o(1)$. In the case of *strong interaction* of a flow around an elongated body, where the angles of inclination of the streamlines are finite, i.e., $v/u_\infty = O(1)$, the flow at a large distance from the ends of the body is near-plane. From the geometry standpoint, this situation corresponds to the theory of high-aspect-ratio wings and the theory of hypersonic bands. A radical difference between the above-considered cases is that it is assumed in the indicated theories that a body being considered has good aerodynamic properties, and, in the stationary analogy, a body of poor aerodynamic quality is considered. In the latter case, the flow separations come out from the interior part of the region being considered to infinity; therefore, the Helmholtz scheme with free boundaries can be used in this case. Since a flow around a poor-aerodynamic-quality body separates on both

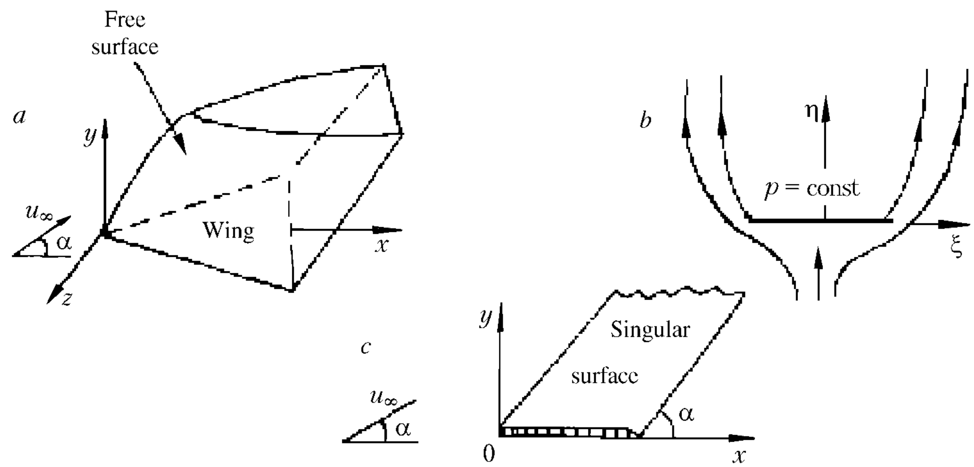


Fig. 2. Scheme of a subsonic flow around a triangular wing of low aspect ratio ($\alpha \sim 1$): a) general view; 2) interior part of the region (Helmholtz scheme, ξ, $\eta \sim 1$); c) exterior part of the region (y, $z \sim 1$).

sides of this body, the near wake downstream of it represents not a continuous vortex sheet but a double tangential-discontinuity surface that, in the external expansion, i.e., in terms of the scale $O(l)$, is reduced to a plane dipole wake.

Instead of (1), we have an internal expansion in terms of the scale $O(\lambda l)$:

$$v\,(x, y, z; \lambda) = V\,(x, \eta, \xi) + O\,(\lambda)\,, \quad w\,(x, y, z; \lambda) = W\,(x, \eta, \xi) + O\,(\lambda)\,,$$

$$p\,(x, y, z; \lambda) = p_0\,(x, \eta, \xi) + O\,(\lambda)\,. \tag{3}$$

The first term of the expansion of the substantive derivative

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + \frac{1}{\lambda}\left(V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial \xi}\right) + O\,(1)$$

is small. In this case,

$$\frac{d}{dt} \approx \frac{1}{\lambda}\left(V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial \xi}\right). \tag{4}$$

In the first approximation, the x dependence does not appear in the equations of motion; it only determines the section in which the flow is considered. Consequently, in the interior part of the region being considered, the functions desired depend on the coordinate x parametrically. Here, the law of plane sections takes the form of a stationary analogy between three-dimensional and two-dimensional flows.

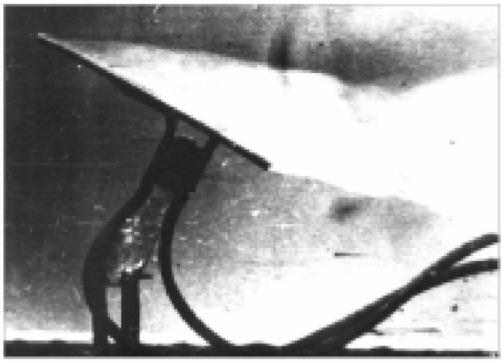
In the main approximation, the longitudinal velocity component depends only on x and ψ — the stream function of the traverse flow:

$$u\,(x, y, z; \lambda) = u_0\,(x, \psi) + O\,(\lambda)\,,$$

where $v = \partial\psi/\partial z$ and $w = -\partial\psi/\partial y$. The subsequent terms in (3) represent λ-additions to the two-dimensional flow.

Figure 2 shows schemes of a separation subsonic flow around a triangular wing of low aspect ratio, positioned at a finite angle of attack $\alpha = O(1)$. Figure 3 shows a flow around a narrow triangular wing (side view). The vortex sheet was visualized with the use of a paint supplied to the edges of the wing. When Fig. 2c and Fig. 3 are compared, it is apparent that they are qualitatively similar to each other.

Approximation (4) differs radically from approximation (2). If in the nonstationary analogy (2) a time-like dependence of the solution on x is true, in the stationary analogy (4) the disturbances are so strong that the dependence



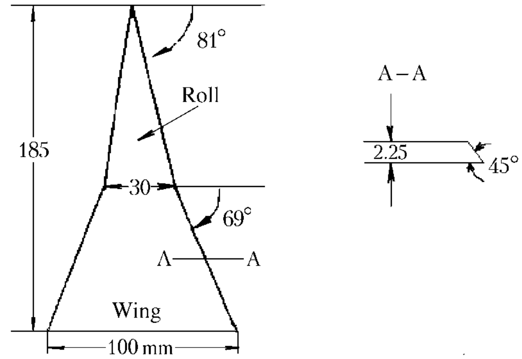


Fig. 3. Photography of a flow around a narrow triangular wing (side view) in a vertical GT-1 hydrotube of the Central Aerohydrodynamics Institute.

Fig. 4. Wing with a bend, a view in plan.

of the solution on x in the main approximation is "frozen." In both cases, it is assumed that the action of the viscous effects is concentrated in the layers having a characteristic thickness $\delta = [\nu / (u_\infty l)]^{1/2} \ll \lambda$. Within the framework of the stationary analogy, we will consider high-rate motions with a characteristic time $O(\lambda)$.

Let us consider two concrete examples.

1. A wing of low aspect ratio ($\lambda \ll 1$) with an angle of attack $\alpha = O(\lambda)$ interacts weakly with a flow around it. The nonstationary analogy applied to the interior part of the region being considered is unusable in the boundary layer, where the disturbance of the transverse flow velocity cannot be considered as small. The boundary layer remains three-dimensional; the nonstationary analogy does not "work" here.

A wing of low aspect ratio with a finite angle of attack $\alpha = O(1)$ interacts strongly with a flow around it. The shape of the body is defined by the equation $f(x, \eta, \xi) = 0$. The condition of its impenetrability $\mathbf{u}\nabla f = 0$ takes the form

$$v\frac{\partial f}{\partial \eta} + w\frac{\partial f}{\partial z} = 0 .$$

An exact solution with a constant longitudinal velocity u = const — a slip flow — is obtained in the case of flow around a cylindrical body, where $\partial f/\partial x = 0$. In the external expansion, the body is reduced to the line $y = x = 0$ and a complex system of vortices move away of its top ($x = 0$) and base ($x = l$).

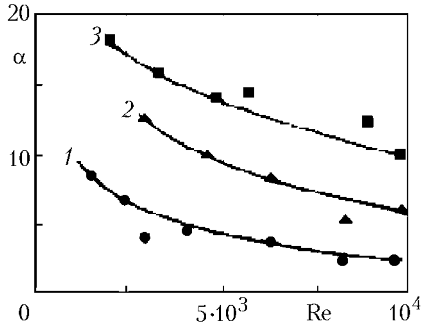
The boundary layer in the stationary analogy is unusual. Relating the velocity $\mathbf{u}$ $\{u, v, w\}$ to u_∞, the pressure to ρu_∞^2, and the coordinates x, y, z to l, we will obtain a flow of an incompressible Newtonian fluid, obeying the Navier–Stokes equation:

$$(u\nabla)\, u + \nabla p = \delta^2 \nabla^2 u\, , \quad (\nabla u) = 0\, . \tag{5}$$

A wing in the form of a plate positioned in the plane $y = 0$ will be considered. On condition that the external expansion is combined with (3), $w = O(1)$. We will determine the characteristic thickness of the boundary layer $\delta\lambda^{1/2}$ from the equation for the z component of the pulse and, from the continuity equation, the ordinal value of the vertical velocity: $v = O(\delta\lambda^{-1/2})$. Thus, the boundary-layer conception is true if $\delta \ll \lambda^{1/2}$. As a result, we have

$$u \approx U\,(x, Y, \xi)\, , \quad v \approx \delta\lambda^{-1/2}\, V\,(x, Y, \xi)\, , \quad w \approx W\,(x, Y, \xi)\, , \quad p \approx P\,(x, Y, \xi)\, ,$$

where $y = \delta/\lambda^{1/2} Y$. In this case, the thickness of the boundary layer is smaller than that in the classical case and the stationary analogy, in accordance with which the x coordinate is "frozen" and the longitudinal and transverse motions are separated, is true:



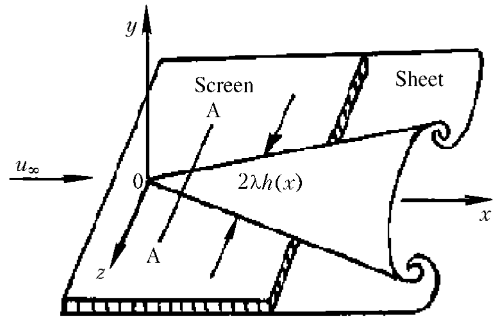


Fig. 5. Flow around a wing with a bend ($\alpha \propto$ Re is a diagram of the flow regimes): 1) α_1(Re); 2) α_2(Re); 3) α_3(Re).

Fig. 6. Triangular cut-out in a screen.

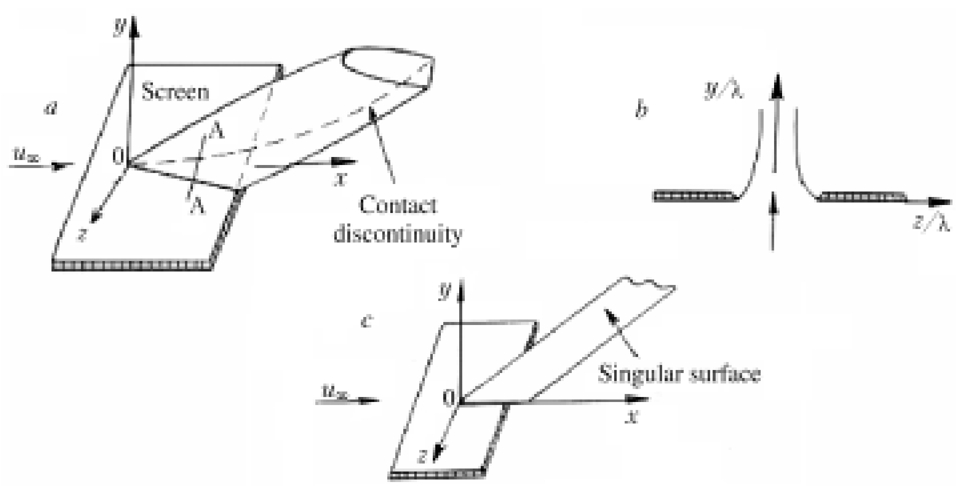


Fig. 7. Flow around a cut-out in the stationary analogy approximation: a) general view; b) interior part of the region; c) exterior part of the region.

$$V\frac{\partial U}{\partial Y}+W\frac{\partial U}{\partial \xi}=\frac{\partial^2 U}{\partial Y^2}\,,\quad \frac{\partial P}{\partial Y}=0\,,\quad V\frac{\partial W}{\partial Y}+W\frac{\partial W}{\partial \xi}+\frac{\partial P}{\partial \xi}=\frac{\partial^2 W}{\partial Y^2}\,,\quad \frac{\partial V}{\partial Y}+\frac{\partial W}{\partial \xi}=0\,.$$

For thin bodies having a complex configuration, the model of a non-separation laminar flow is not valid; in this case, it is necessary to perform a numerical calculation or an experiment. This can be demonstrated by the example of a flow around a wing with a bend (Fig. 4) at small values of Re = δ^{-2}. The experiments were carried out in the above-mentioned hydrotube. At fairly small angles of attack $0 < \alpha < \alpha_1$(Re) there arises a non-separation flow around the wing (see Fig. 5, where the $\alpha \propto$ Re diagram of a flow around the low-aspect-ratio wing–roll aerodynamic combination is presented). When α_1(Re) $< \alpha < \alpha_2$(Re), the partial-separation regime is realized: a vortex is formed at the side edges of the roll and a non-separation flow arises around the wing. In the range α_2(Re) $< \alpha < \alpha_3$(Re), the flow separates from the side edges of the roll and the wing, and, in the region of the bend, the vortex sheet, leaving the edges of the roll, falls to the plane of the wing. Finally, at $\alpha > \alpha_3$(Re) there arises a developed separation flow: the vortex sheet leaving the edges of the roll does not separate from the sheet leaving the side edges of the wing and form an indivisible spiral structure with it.

2. The law of plane sections is also true for narrow flows. Let us consider a stationary noncirculatory axisymmetric flow of an incompressible fluid around a longitudinal cut-out in a plane screen. Such cut-outs are made in wings and in the walls of chemical reactors and wind tunnels.

Figure 6 shows a triangular cut-out in a screen; the origin of coordinates is positioned at the top of the cut-out and the 0y axis is directed along the normal to the screen. The shape of the edges of the cut-out are defined as

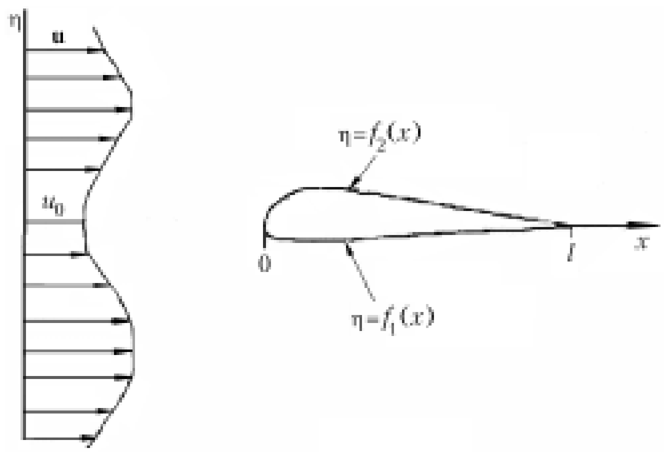


Fig. 8. Scheme of an eddying flow around a thin airfoil (external problem).

$z_0 = \pm\lambda h(x)$, where $\lambda << 1$ is the angle at the top of the cut-out and $h(x)$ is the shape of the cut-out edges in the plane screen. In this problem, the determining parameter is not the angle of attack α but the differential pressure at the screen ε representing the difference between the pressures at $y \to +\infty$ and at $y \to -\infty$.

In the nonstationary analogy approximation, the scheme of the flow includes a mushroom-like separation in the interior part of the region being considered ($y \sim \lambda$, $z \sim \lambda$) and a singular line $y = z = 0$, $0 \le x \le l$, where l is the cut-out length, in the exterior part of the region. The vortex sheets leaving the edges of the cut-out are kept within the interior part of the region.

The nonstationary analogy investigated in [8, 9] can be used in the case where $\varepsilon = O(\lambda^2 \ln \lambda)$. Consequently, the stationary analogy can be used in the case where $O(\lambda^2 \ln \lambda) << \varepsilon << O(1)$. The velocity decomposition is similar to (3). Figure 7a shows the general scheme of the flow around the cut-out being considered in the stationary-analogy approximation; the separation comes out from the interior part of the region to the zone, where the flow is realized by the Helmholtz scheme (Fig. 7b). In the exterior part of the region (Fig. 7c) there is a particular sector of the plane $z = 0$. As in the theory of low-aspect-ratio wings, the boundary layer in the screen remains two-dimensional within the framework of the nonstationary analogy; however, it becomes three-dimensional within the framework of the stationary analogy.

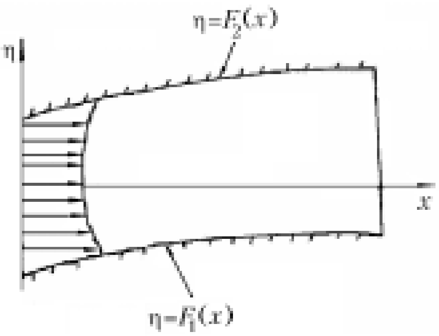
Plane Flow. The nonstationary analogy sets up the correspondence between a three-dimensional hypersonic flow ($M_\infty = O(1/\lambda)$) and a two-dimensional nonstationary flow of a compressible gas as well as the correspondence between a three-dimensional supersonic compressible-gas flow and a two-dimensional nonstationary incompressible-fluid flow. We will show that the problem on a plane incompressible-fluid flow can be solved in the closed form.

Let us consider two problems: the external problem on an eddying flow around a thin airfoil (Fig. 8) with a shape defined by the equation $y_0 = \lambda f(x)$ and the internal problem on a vortex flow in a two-dimensional curvilinear channel (Fig. 9) with a shape defined by the equation $y_0 = \lambda F_{1,2}(x)$, $\lambda << 1$. The internal problem for the case of a compressible-gas flow was considered in [10].

A narrow portion of a smooth curve can be approximated by a linear segment. Therefore, in the main approximation, the velocity distribution over the cross section of a narrow flow becomes linear, which corresponds to a constant vorticity ω. In the case where $\psi = \lambda\Psi(x, \eta) + O(\lambda)$, $\omega(\psi) \approx \omega(\lambda\Psi) \approx \omega(0) = \omega_0$. The value of ω_0 is determined in the Trefftz plane ($x = -\infty$), where the x dependence disappears: in the external problem, the velocity of an undisturbed flow depends on only the coordinate y, and, in the internal problem, the channel has a constant cross section $F(-\infty) = \text{const}$.

The equation for determining the flow function $\psi_{xx} + \psi_{yy} = -\omega(\psi)$ in accordance with the nonstationary analogy becomes *quasi-ordinary*: $\Psi_{\eta\eta} = \Omega$, where $\Omega = -\lambda\omega_0$. The vorticity is significant at $\omega = O(1/\lambda)$. A solution of the quasi-ordinary equation is obtained in the closed form:

$$\Psi = \frac{1}{2}\eta^2\Omega + \eta C_1(x) + C_2(x) \,. \tag{6}$$



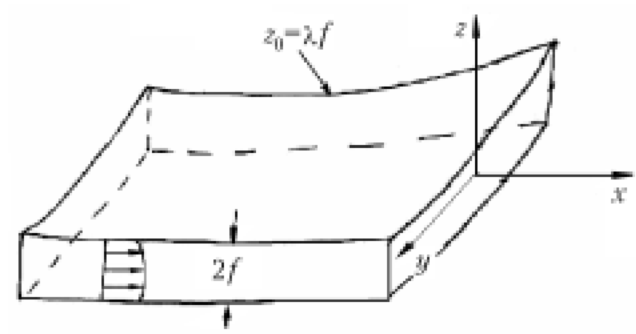


Fig. 9. Scheme of a fluid flow in a two-dimensional channel (internal problem).

Fig. 10. Scheme of a flow in a film.

The functions $C_1(x)$ and $C_2(x)$ are determined from the boundary conditions.

The external problem, because of the symmetry, is solved only in the upper semiplane. The condition $\Psi = 0$ is fulfilled for the following lines: $\eta = 0$ at $x \le 0$, $\eta = f$ at $0 \le x \le l$ and, once again, $\eta = 0$ at $x \ge l$. Then, from (6), at $0 \le x \le l$, we have $C_1 = u_0$ and $C_2 = 0$, where u_0 is the velocity in the zero line of flow in the Trefftz plane. At $0 \le x \le l$, we have $C_1 = -\Omega f$ and $C_2 = \frac{1}{2}\Omega f^2$.

In the internal problem, the unknown functions C_1 and C_2 in (6) are determined on condition that Ψ is constant at the walls of the nozzle: $\Psi = \Psi_{1,2}$ at $\eta = F_{1,2}(x)$. In this case,

$$C_1 = \frac{\Psi_1 - \Psi_2}{F_1 - F_2} - \frac{1}{2}\Omega\,(F_1 + F_2)\,, \quad C_2 = \frac{F_1\Psi_2 - F_2\Psi_1}{F_1 - F_2} + \frac{1}{2}\,\Omega F_1 F_2\,.$$

Flow in a Film. The problem on a nonviscous-liquid flow in a thin layer was solved for the first time by Rayleigh [11]. Let us consider a symmetric noneddying inviscid flow in a film without regard for the gravity force and the surface tension (Fig. 10). It is assumed that the velocity potential $\varphi(x, y, z)$ obeys the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0\,. \tag{7}$$

The thickness of the film is small: $z_0 = \pm\lambda f(x, y)$. At its boundary, the conditions of impenetrability and equality of the pressures on both sides are fufilled:

$$\varphi_z = \lambda f_x \varphi_x + \lambda f_y \varphi_y\,, \quad \varphi_x^2 + \varphi_y^2 + \varphi_z^2 = u_\infty^2\,, \tag{8}$$

where u_∞ is the velocity of the fluid outside the film. In this case, the following expansion is true:

$$\varphi\,(x, y, z; \lambda) = \varphi_0\,(x, y) + \lambda^2 \varphi_1\,(x, y) + o\,(\lambda^2)\,,$$

where $z = \lambda\eta$.

From (7) and the condition of flow symmetry relative to the plane $z = 0$ we find that $\varphi_1 = -\frac{1}{2}\eta^2 \Delta\varphi_0 + a(x, y)$, where $\Delta\varphi_0 = \varphi_{0xx} = \varphi_{0yy}$. From conditions (8) follows the system of two equations for determining the two unknown functions φ_0 and f:

$$\Delta\varphi_0 - f_x \varphi_{0x} - f_y \varphi_{0y} = 0\,, \quad (\Delta\varphi_0)^2 f^2 + \varphi_{0x}^2 + \varphi_{0y}^2 = u_\infty^2\,. \tag{9}$$

Problem (9) can be linearized in the zero approximation: $\varphi_0 = u_\infty x$, $f = f(y)$.

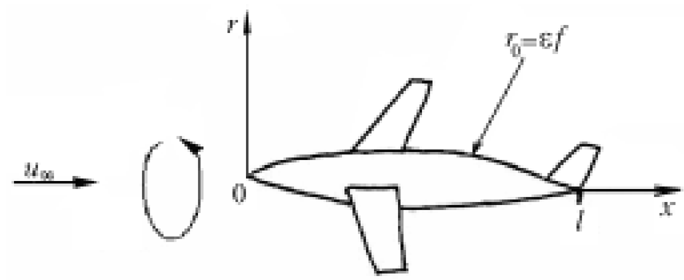


Fig. 11. Scheme of a swirling flow around a thin body.

The above-described asymptotically formalized problem differs radically from the known nonstationary analogy [12] that is based on the coordinate expansion of the solution by Taylor's theorem in a power series of z and, therefore, describes the flow not everywhere over the region but only in the neighborhood of the plane $z = 0$.

Thin Body in a Swirling Flow. A nonviscous liquid flow with velocity components u_x, u_r, and u_θ in the cylindrical coordinate system x, r, θ is defined by the Euler equations

$$\frac{\partial u_x}{\partial x}+\frac{1}{r}\frac{\partial (ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}=0\,,\quad \frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_r\frac{\partial u_x}{\partial r}+\frac{1}{r}u_\theta\frac{\partial u_x}{\partial \theta}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0\,,$$

$$\frac{\partial u_r}{\partial t}+u_x\frac{\partial u_r}{\partial x}+u_r\frac{\partial u_r}{\partial r}+\frac{1}{r}u_\theta\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\,, \tag{10}$$

$$\frac{\partial u_\theta}{\partial t}+u_x\frac{\partial u_\theta}{\partial x}+u_r\frac{\partial u_\theta}{\partial r}+\frac{1}{r}u_\theta\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+\frac{1}{r\rho}\frac{\partial p}{\partial \theta}=0\,.$$

The complete problem can be divided into two problems: the problem on a swirling flow around a stationary body and the problem on a plane-parallel flow around a rotating body. We will consider these problems in order.

1. Let an undisturbed fluid flow around a pointed body (Fig. 11) having a shape $r_0 = \lambda f(x, \theta)$, $\lambda \ll 1$:

$$u_x=u_\infty\,,\quad u_\theta=\tau u_\infty w_0(r)\,,\quad u_r=0\,,\quad p=\rho\int u_0^2\frac{dr}{r}\,,$$

where $u_0(r)$ is an arbitrary function having an integral of u_0^2/r convergent at zero. The condition of impenetrabilty of the body has the form

$$u_r-\lambda u_x\frac{\partial f}{\partial x}-\frac{\lambda}{r}u_\theta\frac{\partial f}{\partial \theta}=0\,. \tag{11}$$

The problem is determined by two dimensional parameters: the elongation of the zone λ and the Squire number τ — the ratio between the rotational velocity and the characteristic translational velocity. (In geophysics, the reciprocal of the Squire number is called the Rossby number). As in the theory of column-like vortices [13, 14], different mathematical models are obtained depending on the ordinal relation between λ and τ.

We now determine the instant the swirling begins to act. Let $\lim_{r\to 0}\omega_0(r) = \alpha r^n = \alpha\lambda^n R^n$, $n > 0$. From the condition of combination of the circular velocity with its undisturbed internal limit follows that the swirling effect is small when $\tau \ll \lambda^{1-n}$ and the asymptotic expansion has the form

$$u_x=1+\lambda^2 u(x,R,\theta)+o(\lambda^2)\,,\quad u_r=\lambda v(x,R,\theta)+o(\lambda)\,,$$
$$u_\theta=\lambda w(x,R,\theta)+o(\lambda)\,,\quad p=\lambda^2 P(x,R,\theta)+o(\lambda^2)\,,\quad r=\lambda R\,. \tag{12}$$

Here, the linear dimensions are related to l, the velocity components are related to u_∞, and the pressure is related to ρu_∞^2. From (10) and (11) we obtain a classical variant of the nonstationary analogy. At $\tau << \lambda^{n-1}$, the swirling has no influence on the main approximation and, at $\tau = \lambda^{n-1}$, the swirling influences this approximation: at infinity, the flow is swirled by the power law. It remains to consider the case of a strong swirling, where $1 >> \varepsilon = \tau\lambda^n >> \lambda$ and the asymptotic expansion has the form

$$u_x = 1 + \lambda\varepsilon u\,(x, R, \theta) + o\,(\lambda\varepsilon)\,,\quad u_r = \varepsilon v\,(x, R, \theta) + o\,(\varepsilon)\,,\quad u_\theta = \varepsilon w\,(x, R, \theta) + o\,(\varepsilon)\,,$$

$$p = \varepsilon^2 P\,(x, R, \theta) + o\,(\varepsilon^2)\,.$$

Here, the stationary analogy is true:

$$\frac{\partial\,(Rv)}{\partial R} + \frac{\partial w}{\partial \theta} = 0\,,\quad v\frac{\partial v}{\partial R} + \frac{1}{R}w\frac{\partial v}{\partial \theta} - \frac{1}{R}w^2 + \frac{\partial P}{\partial R} = 0\,,\quad v\frac{\partial w}{\partial R} + \frac{1}{R}w\frac{\partial w}{\partial \theta} + \frac{1}{R}vw + \frac{1}{R}\frac{\partial P}{\partial \theta} = 0\,.$$

The velocity component u is determined independently of the other components from the equation

$$v\frac{\partial u}{\partial R} + \frac{1}{R}w\frac{\partial u}{\partial \theta} + \frac{\partial P}{\partial x} = 0\,.$$

2. Let a body with a shape $r_0 = \lambda f(x, \theta, t)$ rotates with a constant angular velocity ω in a uniform flow: $u_x = u_\infty$, $u_r = u_\theta = 0$. In the rotating coordinate system $0 \le \eta = \theta + \omega t \le 2\pi$, the flow is steady-state, $r_0 = \lambda f(x, \theta, \eta)$. In this case, (10) is transformed into the system of stationary equations

$$\frac{\partial u_x}{\partial x} + \frac{1}{r}\frac{\partial\,(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \eta} = 0\,,\quad u_x\frac{\partial u_x}{\partial x} + u_r\frac{\partial u_x}{\partial r} + \frac{u_\theta + \omega r}{r}\frac{\partial u_x}{\partial \eta} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,,$$

$$u_x\frac{\partial u_r}{\partial x} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta + \omega r}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0\,,\quad u_x\frac{\partial u_\theta}{\partial x} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta + \omega r}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \eta} = 0\,. \quad (13)$$

The condition of impenetrability of the body has the form

$$u_r - \lambda u_x\frac{\partial f}{\partial x} - \lambda\frac{u_\theta + \omega r}{r}\frac{\partial f}{\partial \theta} = 0\,. \quad (14)$$

At $\tau = \omega l/u_\infty << \lambda$, the swirling, as before, has no influence on the main approximation. At $\tau = \lambda$, the asymptotic expansion (12), the law of plane sections, and the nonstationary analogy are true. In this case, system (13) becomes simplier:

$$\frac{\partial\,(Rv)}{\partial R} + \frac{\partial w}{\partial \eta} = 0\,,\quad \frac{\partial v}{\partial x} + v\frac{\partial v}{\partial R} + \frac{w + R}{R}\frac{\partial v}{\partial \eta} - \frac{1}{R}w^2 + \frac{\partial P}{\partial R} = 0\,,\quad \frac{\partial w}{\partial x} + v\frac{\partial w}{\partial R} + \frac{w + R}{R}\frac{\partial w}{\partial \eta} - \frac{1}{R}vw + \frac{\partial P}{\partial \eta} = 0\,.$$

The impenetrability condition (14) is transformed to the form

$$v - \frac{\partial f}{\partial x} - \frac{w + R}{R}\frac{\partial f}{\partial \eta} = 0\,.$$

When $\tau >> \lambda$, the following asymptotic expansion is valid:

$$u_x = 1 + \tau^2 u\,(x, R, \eta) + o\,(\tau^2)\,,\quad u_r = \tau v\,(x, R, \eta) + o\,(\tau)\,,\quad u_\theta = \tau w\,(x, R, \eta) + o\,(\tau)\,,$$

$$p = \tau^2 P\,(x, R, \eta) + o\,(\tau^2)\,.$$

From (13) it follows that

$$\frac{\partial (Rv)}{\partial R}+\frac{\partial w}{\partial \eta}=0\ ,\quad v\frac{\partial v}{\partial R}+\frac{w+R}{R}\frac{\partial v}{\partial \eta}-\frac{1}{R}w^{2}+\frac{\partial P}{\partial R}=0\ ,\quad v\frac{\partial w}{\partial R}+\frac{w+R}{R}\frac{\partial w}{\partial \eta}-\frac{1}{R}vw+\frac{\partial P}{\partial \eta}=0\ .$$

Thus, in the case where $\tau >> \lambda$, the stationary analogy is true and the impenetrability condition (14) becomes two-dimensional:

$$v-\frac{w+R}{R}\frac{\partial f}{\partial \eta}=0\ .$$

Conclusions. The law of plane sections, which is not only of independent importance but also represents a standard for numerical calculations, is used in the form of nonstationary and stationary analogies. In the nonstationary-analogy regime, a separation flow is similar to the Markov process: only the effect of memory (of the vortex sheet shape) in the section being considered x = const is taken into account. Due to the Markov effect, the problem with initial data is reliable in the interior part of the region being considered.

In the present work, the stationary analogy, in accordance with which the flows around the cross sections in the disturbed zone are independent, is applied to the theory of thin bodies and to the problem on a flow around a narrow longitudinal cut-out in a screen. A common property of these problems is the existence of a series expansion parameter — the dimensional thickness of the zone λ. The second series expansion parameters used in these problems are different: in the theory of low-aspect-ratio wings, this parameter is the angle of attack α, and, in the problem on a flow through a slot, the second parameter is the differential pressure ε. Depending on their ordinal values in relation to λ ($\alpha \propto \lambda$, $\varepsilon \propto \lambda$) a stationary analogy or a nonstationary one are obtained. The nonstationary analogy in the problem on a flow around a thin body positioned at a definite angle of attack is incorrect [6].

If the law of plane sections is fulfilled within the framework of the nonstationary analogy for an observer moving along the axis of a zone with a velocity of an undisturbed flow, this law is fulfilled within the framework of the stationary analogy for an immovable observer.

In the stationary-analogy regime, a separation leaving the edges of a wing or a slot (in the form of vortex sheets in a one-phase ideal liquid) in the scale of the interior part of the region, equal to λ, goes to infinity, i.e., leaves the region. In the scale of the exterior part of the region, the whole separation (interior) part of the region, including the wake, is reduced to a singular line.

In the approximation of both the nonstationary and stationary analogies, a three-layer flow including a boundary layer, an interior nonviscous part and an exterior nonviscous part are considered.

NOTATION

l, characteristic dimension; M, Mach number; p, pressure; Re, Reynolds number; t, time; $\mathbf{u}$, velocity of a fluid; u, v, w, velocity components along the Cartesian axes x, y, z; u_x, u_r, u_θ, velocity components along the axes of the cylindrical coordinate system x, r, θ; α, angle of attack; δ, characteristic thickness of the boundary layer; λ, series expansion parameter characterizing the dimensionless thickness of a zone or the angle of inclination of the streamlines; ∇, gradient operator; ξ, η, subsidiary coordinates; ε, differential pressure; ν, coefficient of kinematic viscosity; ρ, density of the fluid; τ, Skwayer number; φ, velocity potential; ψ, flow function; ω, vorticity and angular velocity. Subscripts: ∞, undisturbed flow.

REFERENCES

1. W. G. Hayes and R. F. Probstein, *Hypersonic Flow Theory* [Russian translation], IL, Moscow (1962).
2. H. Ashley and M. Landahl, *Aerodynamics of Wings and Bodies* [Russian translation], Mashinostroenie, Moscow (1969).
3. J. D. Cole, *Perturbation Methods in Applied Mathematics* [Russian translation], Mir, Moscow (1972).
4. V. V. Lunev, *Hypersonic Aerodynamics* [in Russian], Mashinostroenie, Moscow (1975).

5. J. D. Cole and L. P. Cook, *Transonic Aerodynamics* [Russian translation], Mir, Moscow (1980).
6. V. V. Sychev, Spatial hypersonic gas flows near thin bodies at large angles of attack, *Prikl. Mat. Mekh.*, **24**, Issue 2, 205–212 (1960).
7. H. K. Cheng, Aerodynamics of a rectangular plate with vortex separation in supersonic flow, *JAS*, **22**, No. 4, 217–226 (1955).
8. S. K. Betyaev, Formation of a jet in a nonstationary ideal fluid flowing from a slot, *Prikl. Mat. Mekh.*, **45**, Issue 6, 1032–1040 (1981).
9. S. K. Betyaev, On the theory of elongated separation zones, *Prikl. Mekh. Tekh. Fiz.*, No. 5, 126–133 (1987).
10. A. A. Nikol'skii, *Theoretical Investigations on Fluid Mechanics* [in Russian], TsAGI, Moscow (1981).
11. J. Rayleigh, *The Theory of Sound* [Russian translation], Vols. 1–2, GITTL, Moscow (1955).
12. M. S. Borgas and E. O. Tuck, The slender water jet, *J. Fluid Mech.*, **118**, 379–391 (1982).
13. S. K. Betyaev, Mathematical simulation of a column-like vortex, *Teor. Osn. Khim. Tekhnol.*, **34**, No. 5, 473–477 (2000).
14. S. K. Betyaev, Mathematical simulation of a nonaxisymmetric column-like vortex, *Teor. Osn. Khim. Tekhnol.*, **36**, No. 2, 124–129 (2002).